Discrete log, factoring, and proofs of quantumness in 2026

Greg Meyer

a.k.a. Greg Kahanamoku-Meyer

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Post-quantum cryptography migration timelines are shifting

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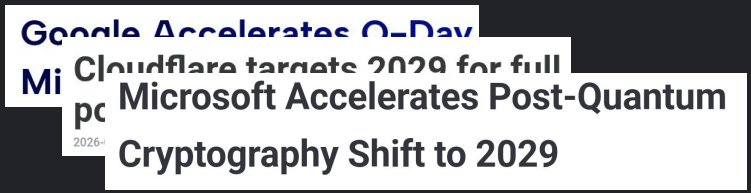

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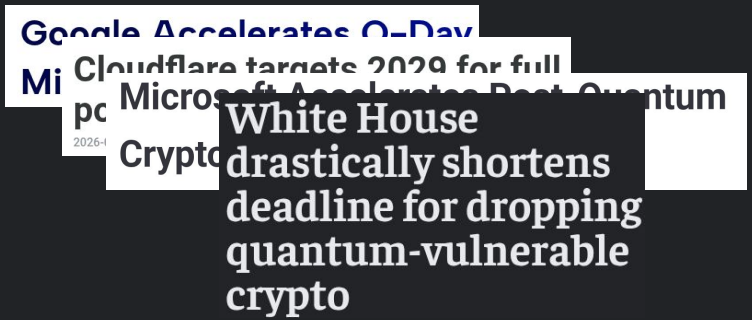

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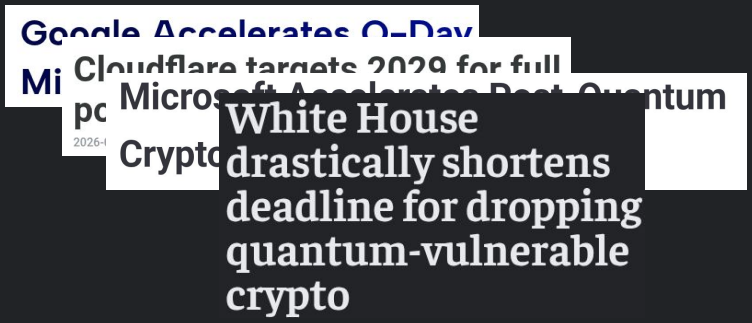

How did this happen?

Structure of this talk

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In this talk:

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- Heuristics and assumptions that seemed reasonable, but have not held up
- **Setting:** algorithms for cryptography
  - Factoring and discrete log
  - Proofs of quantumness
- How should this inform our work moving forward?

Overview

Assumption: factoring is the right thing to focus on

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Comparing factoring and discrete log

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Is the focus on factoring appropriate?

Comparison: use in modern classical cryptography

Example: digital signature schemes

Data: Cloudflare blog (blog.cloudflare.com/ml-dsa-will-have-to-do/)

Scheme	Ed25519	RSA-2048
Vuln. problem	EC discrete log	factoring
Public key size (bytes)	32	272
Signature size (bytes)	64	256
Time to sign (arb. units)	0.15	80
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Schemes relying on the hardness of **elliptic curve discrete log** are used more widely on the internet.

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Note: these are **logical** costs, in the abstract circuit model.

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Elliptic curve discrete log

Given $x = [d]g$, find d.

For curve `secp256k1`:

- Qubits: ~ 1200
- Toffolis: $\sim 2^{26}$

Babbush et al. ’26 (arXiv:2603.28846)

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Elliptic curve discrete log has cheaper quantum circuits (as of now).

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... but there's even another reason to focus on discrete log...

Let’s see who you really are...

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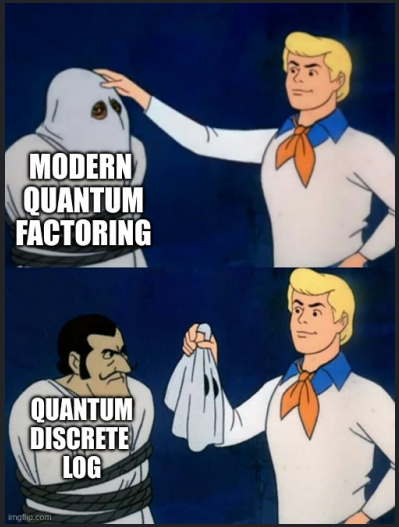

Factoring RSA integers via discrete logarithm

Foundational paper: Ekerå and Håstad ’17 (arXiv:1702.00249)

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Consider an integer $N = pq$.

For the vast majority of positive integers $g < N$:

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So, we can factor via a discrete log. **Is this helpful?**

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Factor by finding period of $f(x) = a^x \bmod N$.

Fact: period has size $O(N)$.

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Ekerå-Håstad ’17

Factor by taking the discrete log of $g^{N+1} \equiv g^{p+q} \pmod{N}$

Fact: period has size $O(\sqrt{N})$.

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Summary: nowadays, all RSA factoring circuits...

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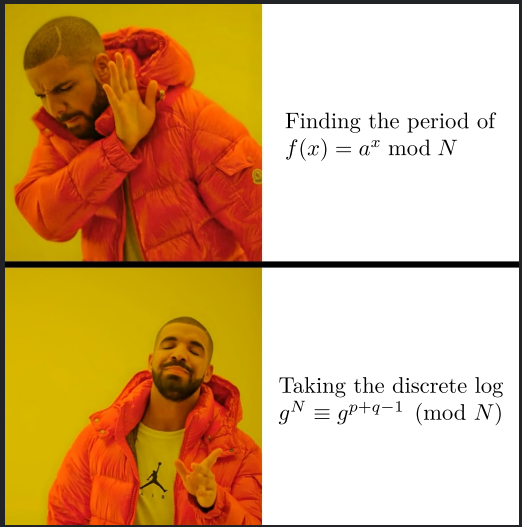

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Reality:

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Assumption: factoring is the thing to focus on

Reality:

- Elliptic curve discrete log is **more widely relied upon** than factoring
- Elliptic curve discrete log has **cheaper quantum circuits** than factoring
- Modern circuits for RSA factoring are **discrete log** circuits anyway

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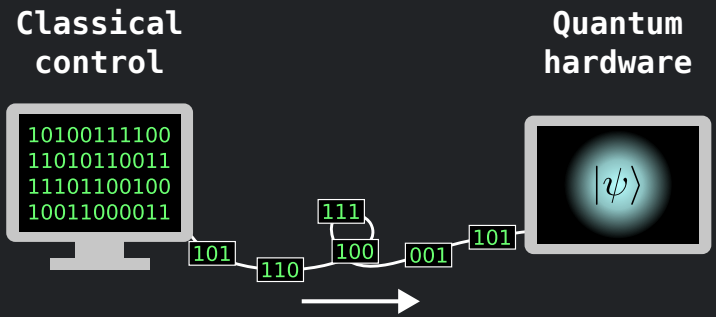

If we input an n-bit value, we must need n qubits, right? Maybe not!

Streaming algorithms

Typical setting

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Streaming as a limitation

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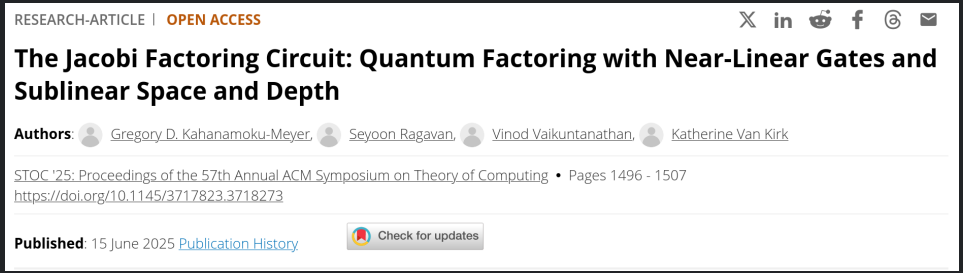
Streaming as a limitation	Streaming as a tool
"You only get to see m bits of the n-bit input at a time"	"Find an algorithm that only needs to see m bits of the n-bit input at a time!"

Two quantum algorithms that do this:

1. Jacobi factoring
2. New discrete log/RSA factoring algorithms

Jacobi factoring



RESEARCH-ARTICLE | OPEN ACCESS

The Jacobi Factoring Circuit: Quantum Factoring with Near-Linear Gates and Sublinear Space and Depth

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STOC '25: Proceedings of the 57th Annual ACM Symposium on Theory of Computing • Pages 1496 - 1507
https://doi.org/10.1145/3717823.3718273

Published: 15 June 2025 Publication History

Check for updates

Quantum algorithm for factoring integers of the **special form** $N = P^2Q$.

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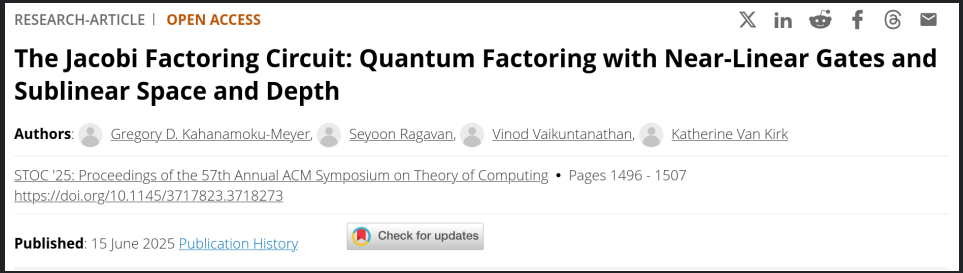

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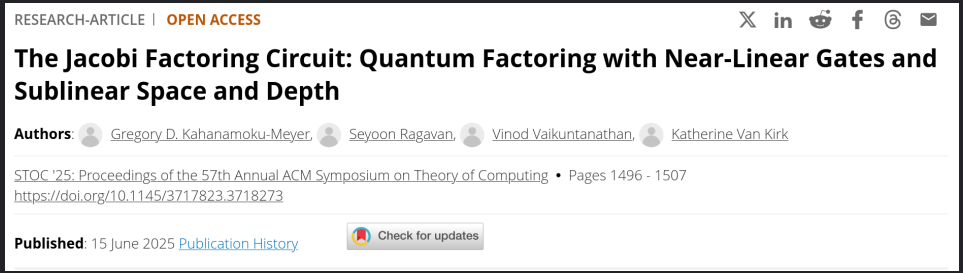

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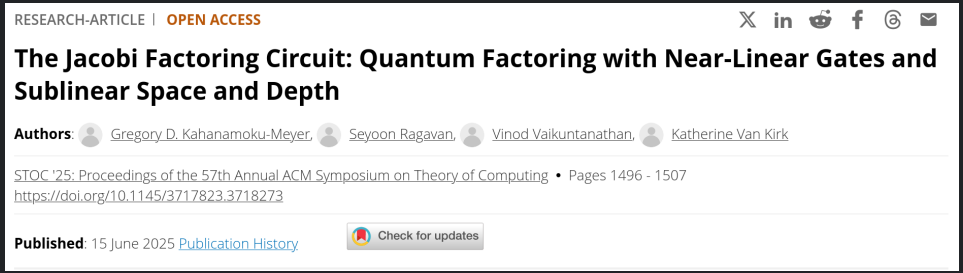

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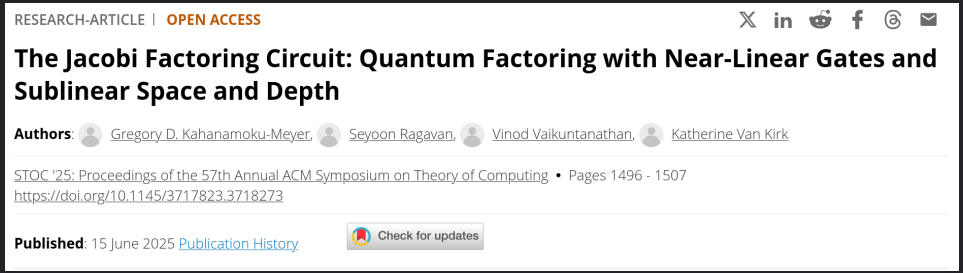

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Let $n = \lceil \log N \rceil$ and $m = \lceil \log Q \rceil = O(n^{2/3})$.
(smallest m that maintains classical hardness)

- Gates: $\widetilde{O}(n)$
- Qubits: $\widetilde{O}(n^{2/3})$
- Depth: $\widetilde{O}(n^{2/3})$

Jacobi factoring: how does it work?

Observation

The *Jacobi symbol*

$$f(x) = \left(\frac{x}{N}\right)$$

is periodic with period Q
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But how do we compute the Jacobi symbol?

Computing the Jacobi symbol in low space

Goal: circuit to implement $\sum_x |x\rangle |0\rangle \to \sum_x |x\rangle |\left(\frac{x}{N}\right)\rangle$

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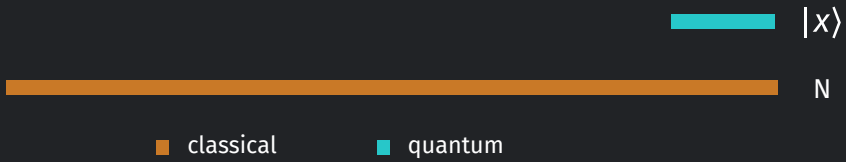
Existing circuits

- Take two quantum inputs
- Both **gate** and **qubit** count linear in the larger input
- Using them directly would yield $O(n)$ qubit count 😢

Streaming classical bits to a quantum circuit

Key idea: using properties of the Jacobi symbol, find an "equivalent" m-bit value $|N'\rangle$ such that we only need to compute $\left(\frac{x}{N'}\right)$



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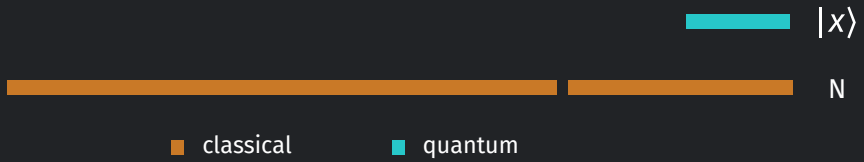

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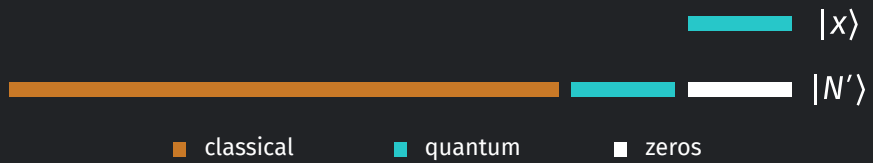

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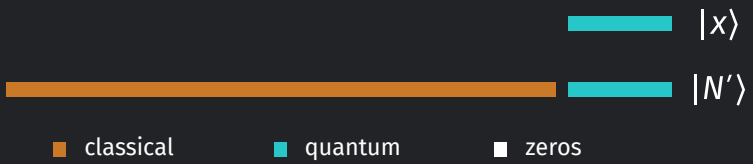

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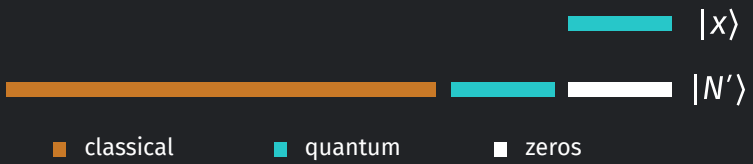

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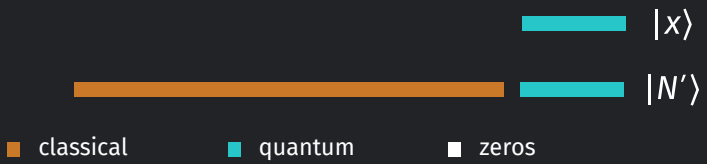
Key idea: using properties of the Jacobi symbol, find an "equivalent" m-bit value $|N'\rangle$ such that we only need to compute $\left(\frac{x}{N'}\right)$



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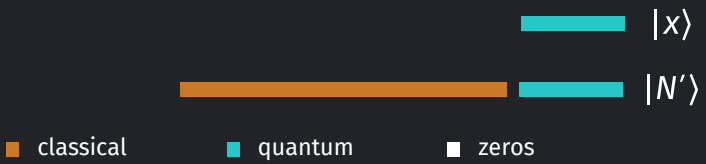

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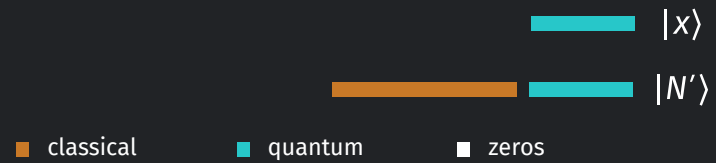

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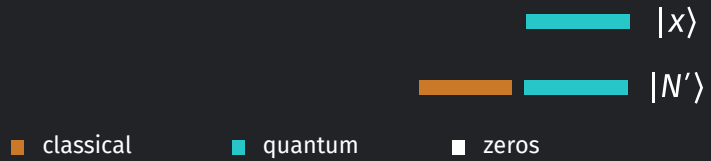

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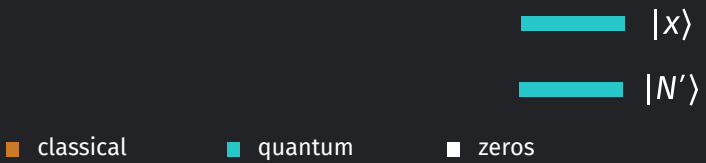
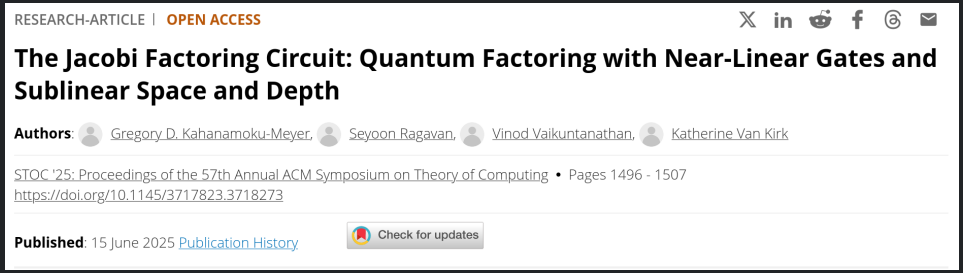


Now with two length-m quantum inputs,
standard circuits for $\left(\frac{x}{N'}\right)$ have depth and qubits $\widetilde{O}(m)$

Summary: Jacobi factoring



RESEARCH-ARTICLE | OPEN ACCESS

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Authors: Gregory D. Kahanamoku-Meyer, Seyoon Ragavan, Vinod Vaikuntanathan, Katherine Van Kirk

STOC '25: Proceedings of the 57th Annual ACM Symposium on Theory of Computing • Pages 1496 - 1507
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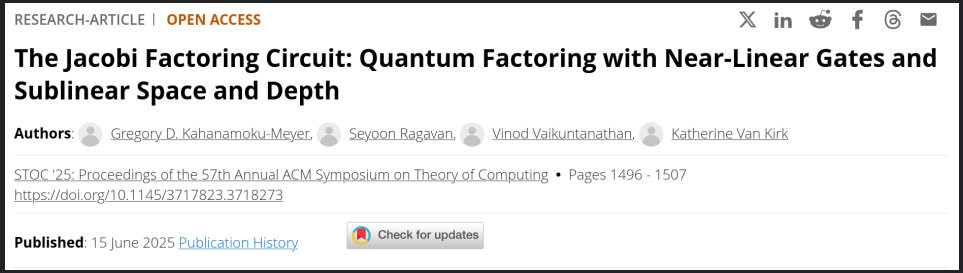

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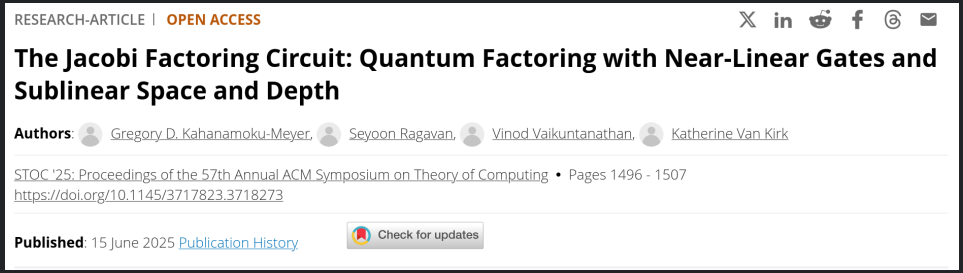

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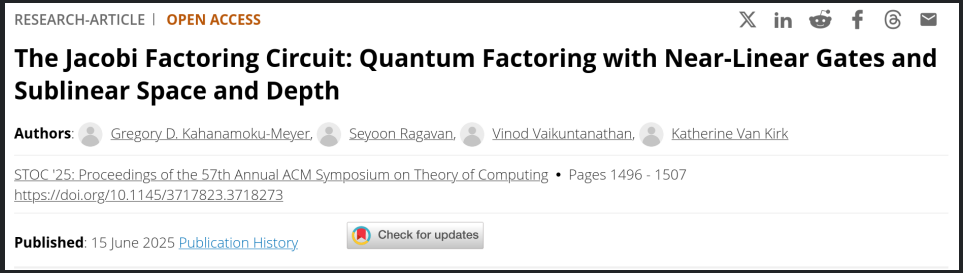

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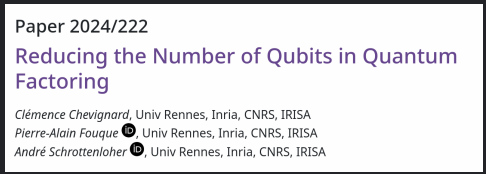
What can we do instead?

Very space-efficient discrete log modulo N (and thus RSA factoring)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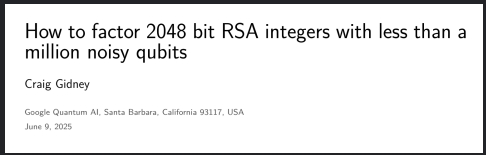
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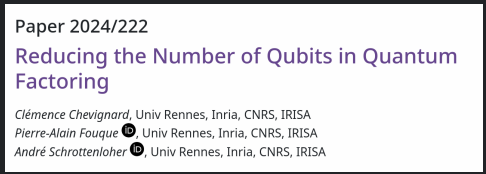
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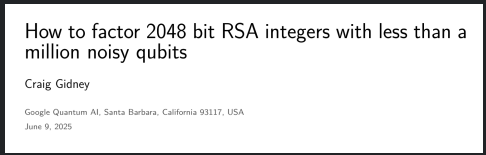
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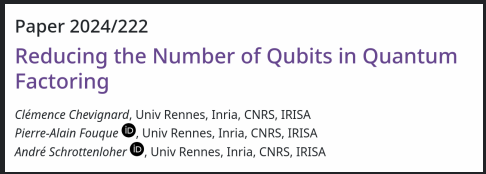
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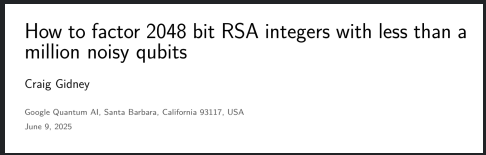
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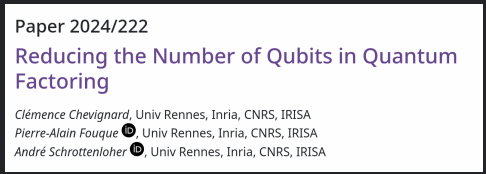
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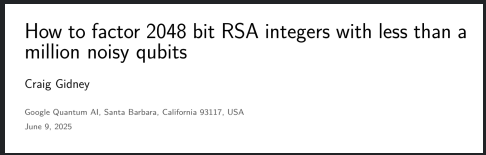
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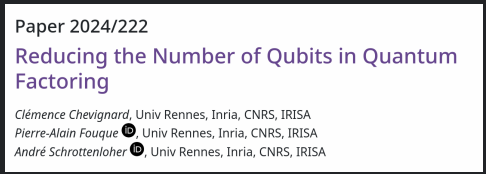
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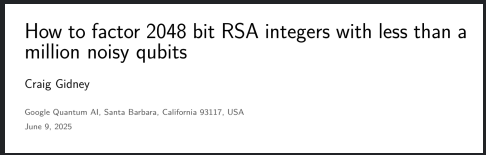
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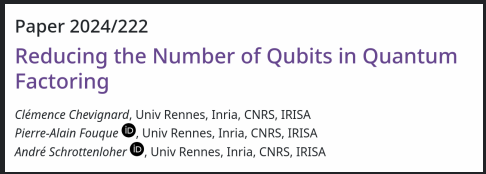
Corollary: Factoring n-bit RSA integers with $\sim n/2$ qubits.

Public service announcement: beware of overly focusing on factoring!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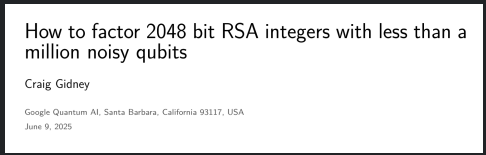
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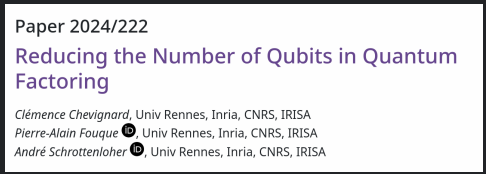
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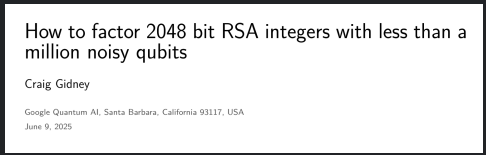
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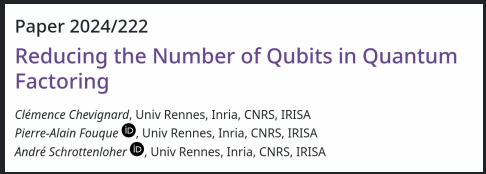
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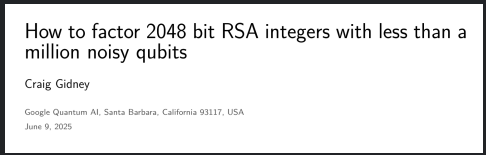
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Proof of quantumness: forget about factoring...

What if we set up a discrete log problem in which $\log d$ is as small as possible?

Recall: costs from beginning of talk

Note: these are **logical** costs, in the abstract circuit model.

Elliptic curve discrete log

Given $x = [d]g$, find d.

For curve `secp256k1`:

- Qubits: ~ 1200
- Toffolis: $\sim 2^{26}$

Babbush et al. '26 (arXiv:2603.28846)
Schrottenloher '26 (arXiv:2606.02235)

RSA integer factorization

Given $N = pq$, find p and q.

For RSA-2048:

- Qubits: ~ 1400
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Finite field discrete log

Given $x \equiv g^d \pmod N$ find d.

For 2048-bit N and 225-bit d:

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For 2048-bit N and 225-bit d:

- Qubits: ~ 350
- Toffolis: $\sim 2^{26}$

(note: preliminary numbers!)

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Good thing finite field discrete log isn't really used in practice, right?

Finite-field discrete log is used in standardized cryptography

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RFC **7919:** Negotiated Finite Field Diffie-Hellman Ephemeral Parameters for Transport Layer Security (TLS)

D. Gillmor

PROPOSED STANDARD

Internet Engineering Task Force (IETF) — D. Gillmor
Request for Comments: 7919 — ACLU
Updates: 2246, 4346, 4492, 5246 — August 2016
Category: Standards Track
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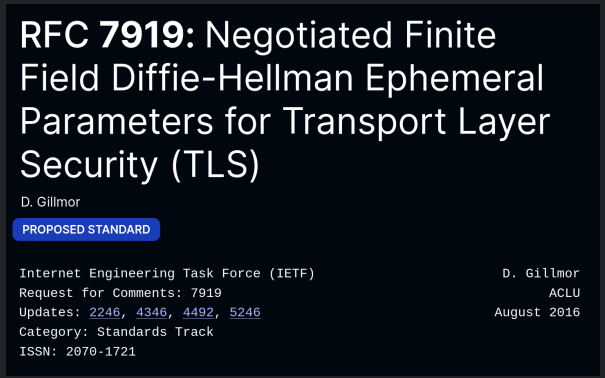

OK, but surely the parameters aren’t set so the exponent is small, right?

The parameters are absolutely set like that

RFC 7919:

“Peers using [a 2048-bit modulus] that want to optimize their key exchange with a short exponent (Section 5.2) should choose a secret key of at least 225 bits.”

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“Peers using [a 2048-bit modulus] that want to optimize their key exchange with a short exponent (Section 5.2) should choose a secret key of at least 225 bits.”

Comparing costs

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All of these are relevant for real-world cryptography!!

Summary

Assumption: input length is a lower bound on qubit count

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Two number-theory counterexamples:

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Key insight: *streaming*—the qubits never need to hold the entire input at once

Overview

Assumption: factoring is the right thing to focus on

Assumption: input length is a lower bound on qubit count

Assumption: logical qubit count is what matters

Assumption: quantum circuits must be (mostly) reversible

Assumption: breaking cryptography is a long way off

Metrics for cost

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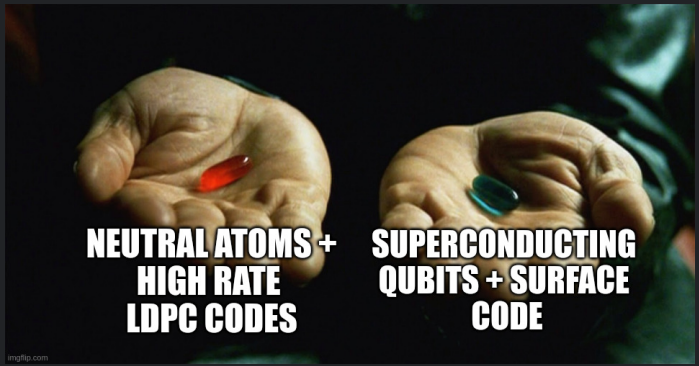

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Take a second look at old constructions under new heuristics?

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Using measurements to uncompute

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when f has many irreversible steps?

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Is it possible to do better?

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[Bennett ’89] introduced pebble games:

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x x_1 x_2 x_3 $f(x)$

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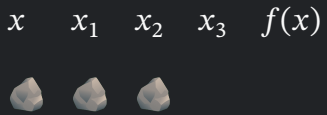

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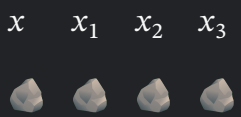

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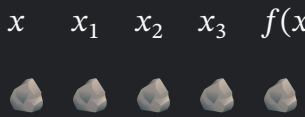

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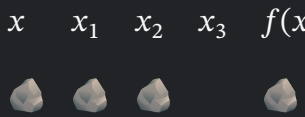

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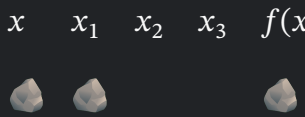

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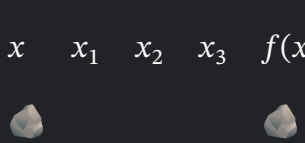

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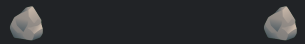

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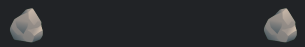

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Time cost (# steps): $2k - 1$ (optimal) 😃

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Space (# pebbles): $O(k)$ 😭

Using less space: a recursive strategy

1. Pebble game from start to $k/2$

x $\cdots$ $x_{k/2}$ $\cdots$ $f(x)$

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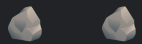

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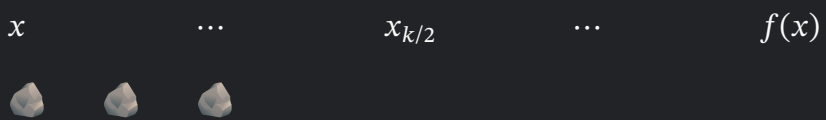

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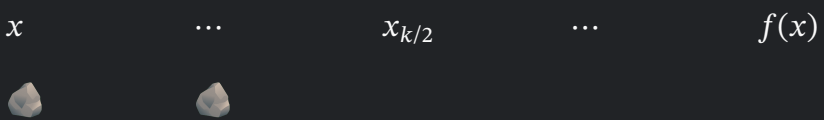

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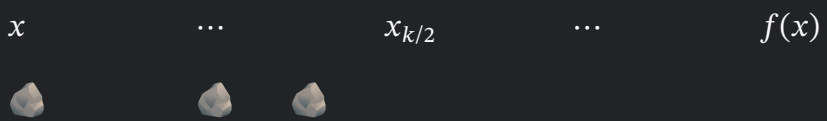

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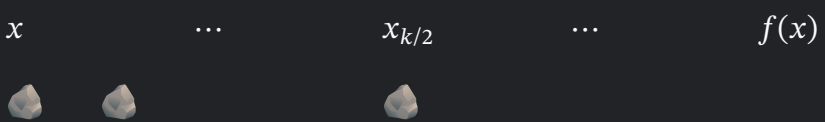

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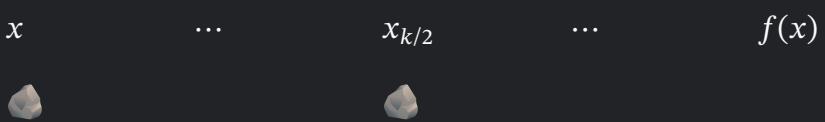

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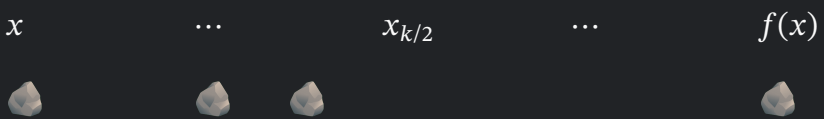

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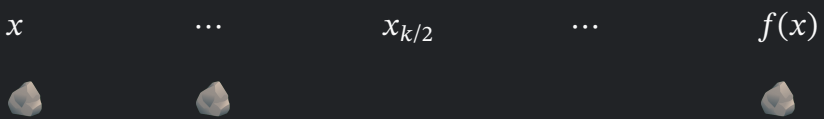

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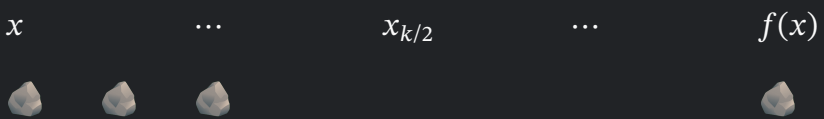

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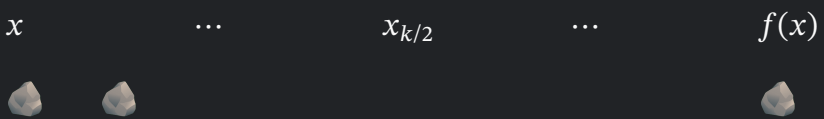

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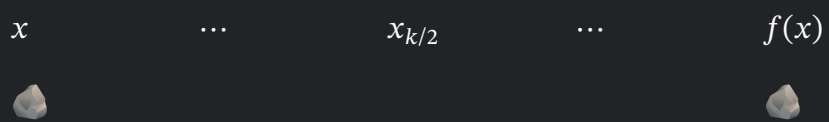

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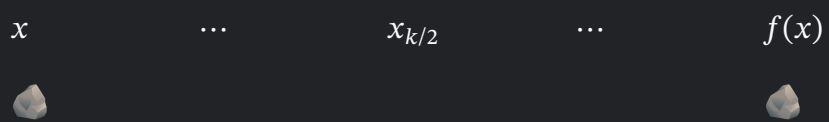

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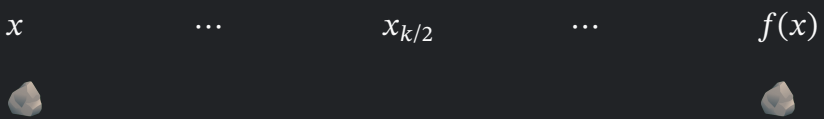

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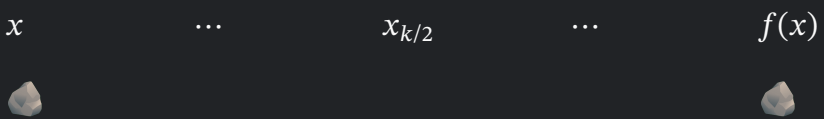

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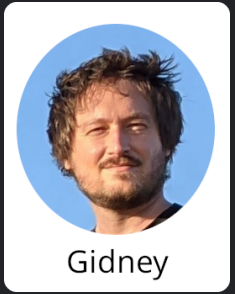
Time cost (# steps): $O(k^{\log_2 3}) \approx O(k^{1.58\cdots})$ steps 😢
Generalization: space $O(2^{1/\epsilon} \log k)$, time $O(k^{1+\epsilon})$ 😐

Spookiness: a quantum resource

“Spooky Pebble Games and Irreversible Uncomputation”
algassert.com/post/1905

Spookiness: a quantum resource



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Using measurement to uncompute

Given an intermediate value

$$|x_i\rangle$$

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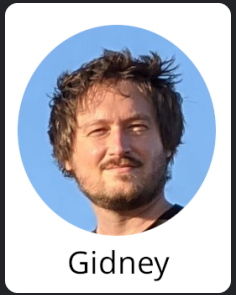

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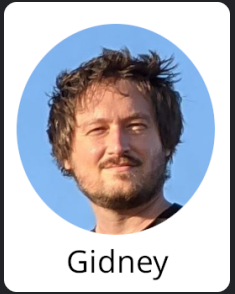

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where d is *classically-known*.

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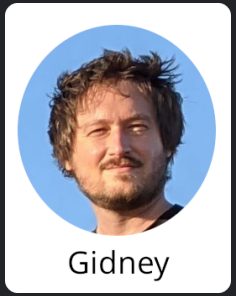

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We've turned $|x_i\rangle$ into a ghost!

Spookiness: a quantum resource

Ghosting: replacing $|x_i\rangle$ with phase error $(-1)^{d \cdot x_i}$ for *classically-known* d

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- can ghost a pebble at any time—but must exorcise later
- exorcism is free when x_i has a pebble

The power of spooky pebbling

Recursive strategy

1. Pebble game from start to $k/2$
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x $\cdots$ $x_{k/2}$ $\cdots$ $f(x)$

The power of spooky pebbling

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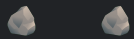

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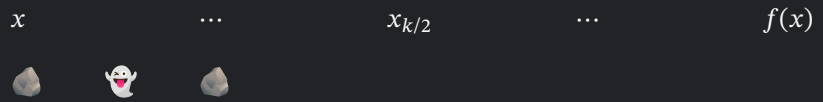

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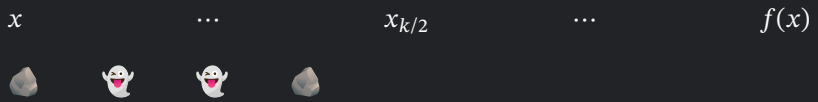

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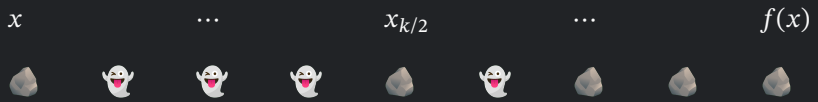

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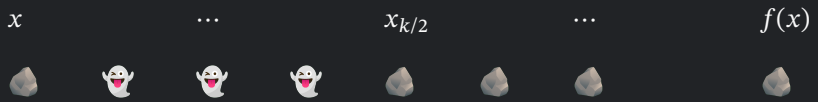

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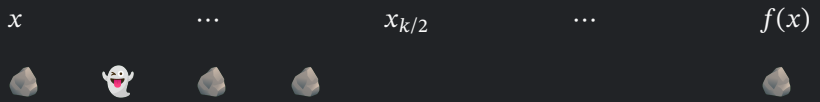

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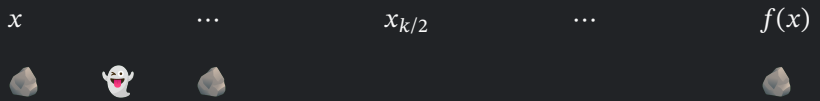

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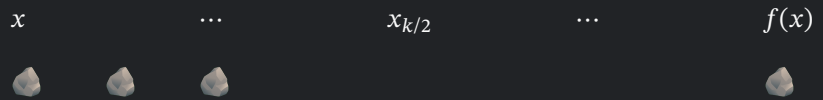

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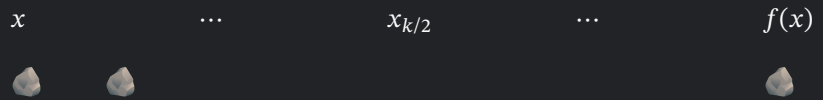

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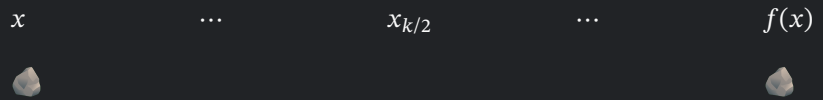

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x $\cdots$ $x_{k/2}$ $\cdots$ $f(x)$


Space: $O(\log k)$ pebbles 😃 **Time cost (# steps):**

The power of spooky pebbling

Spooky recursive strategy

1. *Blast* from start to $k/2$
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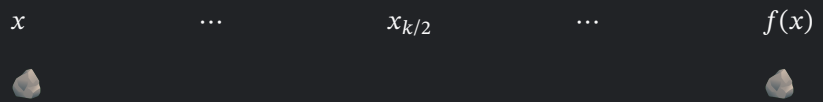

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The power of spooky pebbling

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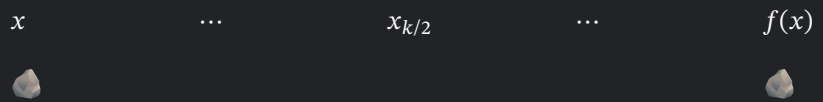

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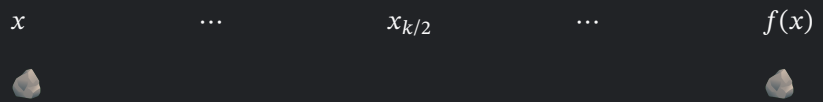

Space: $O(\log k)$ pebbles 😃

Time cost (# steps): $T(k) = O(k) + 2T(k/2)$

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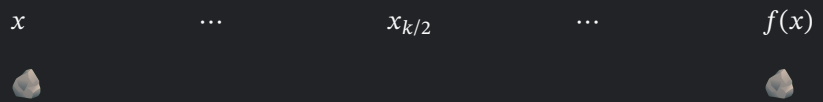

Space: $O(\log k)$ pebbles 😃

Time cost (# steps): $O(k \log k)$ steps 🤩

The power of spooky pebbling

Algorithm	Parallel	Spooky	Space	Depth
Trivial	N	N	k	$2k-1$
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The power of spooky pebbling

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Is the best of both worlds possible?

The power of spooky pebbling

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KMRV '25	Y	Y	$O(\log k)$	$2k-1$

Is the best of both worlds possible?
Yes! Kahanamoku-Meyer, Ragavan, Van Kirk (arXiv:2510.08432)

Example: an optimal parallel spooky pebble game for $k = 12$

x x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} $f(x)$

Step number: 0

Max. pebble count: 1

Example: an optimal parallel spooky pebble game for $k = 12$

x x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} $f(x)$

Step number: 1

Max. pebble count: 2

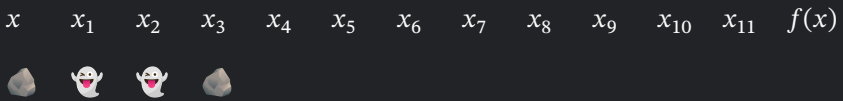
Example: an optimal parallel spooky pebble game for $k = 12$

x x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} $f(x)$

Step number: 2

Max. pebble count: 2

Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 3

Max. pebble count: 2

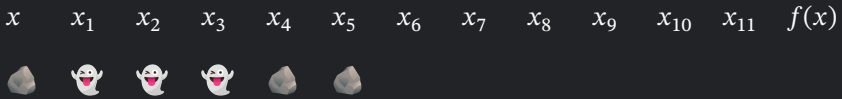
Example: an optimal parallel spooky pebble game for $k = 12$

x	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	$f(x)$
🪨	👻	👻	👻	🪨								

Step number: 4

Max. pebble count: 2

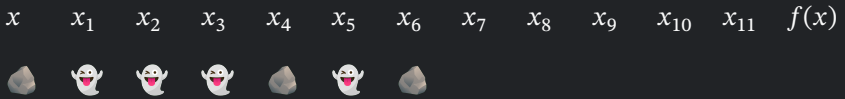
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 5

Max. pebble count: 3

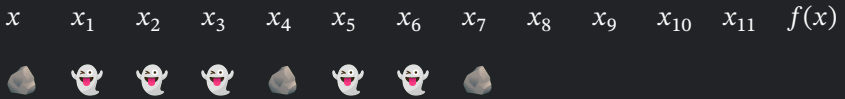
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 6

Max. pebble count: 3

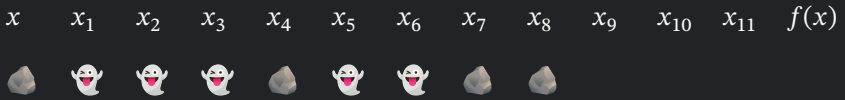
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 7

Max. pebble count: 3

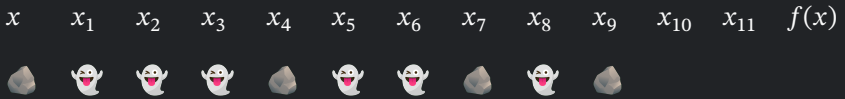
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 8

Max. pebble count: 4

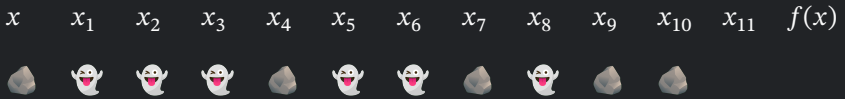
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 9

Max. pebble count: 4

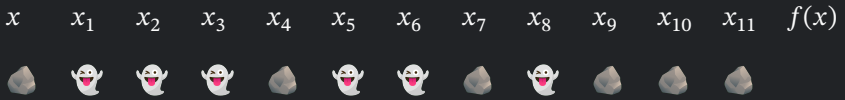
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 10

Max. pebble count: 5

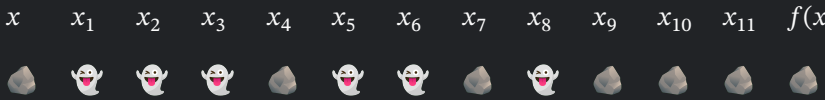
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 11

Max. pebble count: 6

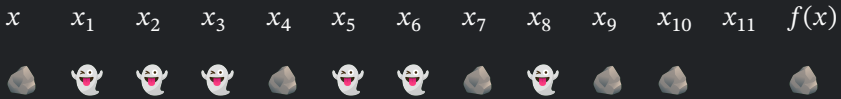
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 12

Max. pebble count: 7

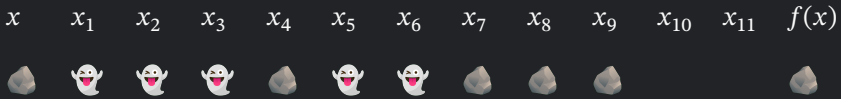
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 13

Max. pebble count: 7

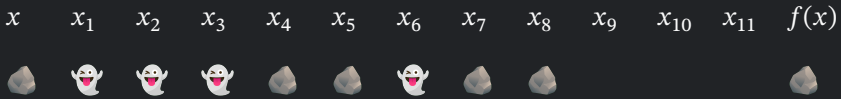
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 14

Max. pebble count: 7

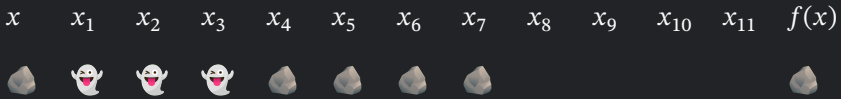
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 15

Max. pebble count: 7

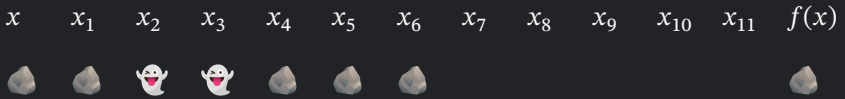
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 16

Max. pebble count: 7

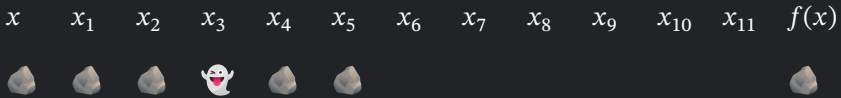
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 17

Max. pebble count: 7

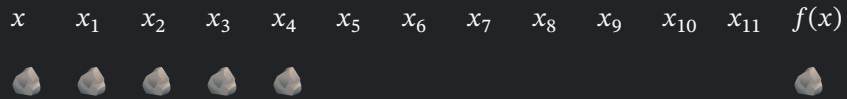
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 18

Max. pebble count: 7

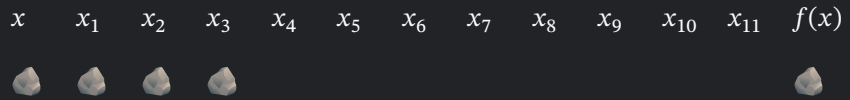
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 19

Max. pebble count: 7

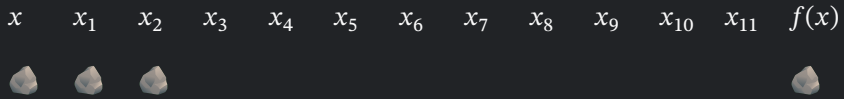
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 20

Max. pebble count: 7

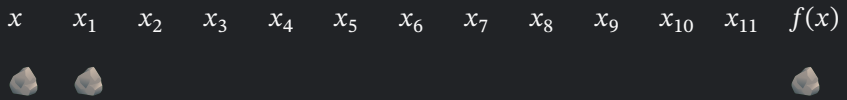
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 21

Max. pebble count: 7

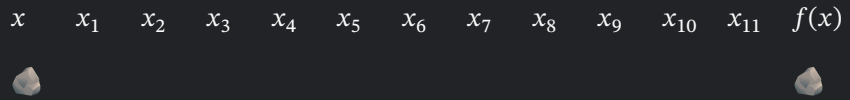
Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 22

Max. pebble count: 7

Example: an optimal parallel spooky pebble game for $k = 12$



Step number: 23

Max. pebble count: 7

Example: an optimal parallel spooky pebble game for $k = 12$

x x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} $f(x)$

Step number: $23 = 2k - 1$, which is optimal 🤩

Max. pebble count: 7

Summary

Assumption: quantum circuits must be (mostly) reversible

Reality:

- **Measurement** is useful at the logical level for much more than just readout

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- Judicious use of intermediate measurements can **dramatically lower circuit sizes**

Some reading:

- Measurement-based uncomputation makes addition much cheaper
  - Reducing T count to $4n$: Gidney ’17 (arXiv:1709.06648)
  - Constant number of ancillas for classical-quantum addition: Gidney ’25 (arXiv:2507.23079)
- Superposition masking: Jaques + Gidney ’20 (arXiv:2008.04577)

Overview

Assumption: factoring is the right thing to focus on

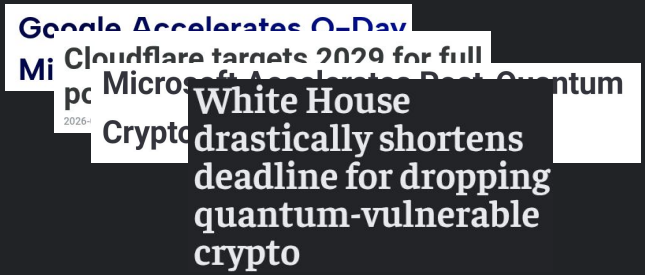
Assumption: input length is a lower bound on qubit count

Assumption: logical qubit count is what matters

Assumption: quantum circuits must be (mostly) reversible

Assumption: breaking cryptography is a long way off

Quantum-vulnerable cryptography is closer than ever to being broken



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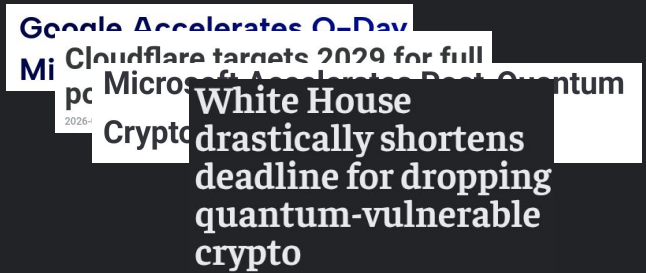
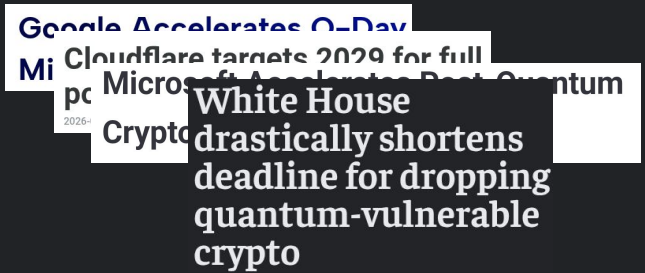


- Sudden algorithmic improvements over the past few years have exceeded all expectations

Quantum-vulnerable cryptography is closer than ever to being broken



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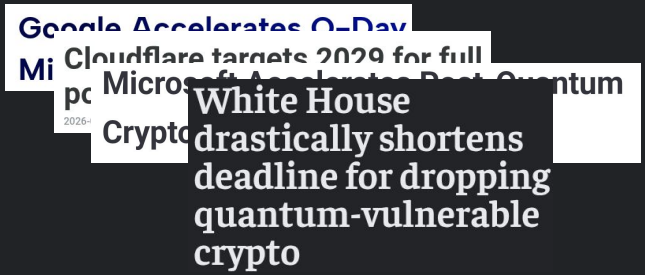

- Sudden algorithmic improvements over the past few years have exceeded all expectations
- Rapidly evolving changes in expected *architecture* and thus in logical circuit *cost metrics*
- These changes have not only shifted the timeline, but also increased the **uncertainty**

Looking forward

What assumptions are about to be broken?
Let’s think about it this week!

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POSTDOCTORAL RESEARCH
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SIMONS INSTITUTE
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Thank you!

Email: gkm@mit.edu
Website: https://gmeyer.net/

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Backup